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## CERTAIN FACTORS OF THE GROUP DETERMINANT.

By WILLIAM BENJAMIN FITE.

So far as I know the following method for finding factors of a group determinant is new.

To the  $n$  operators  $s_1, s_2, \dots, s_n$  of a group  $G$  of finite order we make correspond  $n$  independent variables  $x_{s_1}, x_{s_2}, \dots, x_{s_n}$ , such that  $x_{s_k} \equiv x_{s_i s_j}$  if  $s_k = s_i s_j$ . With these variables we form the following determinant, which is called the group determinant of  $G$ :

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1 s_1^{-1}} & x_{s_1 s_2^{-1}} & \dots & x_{s_1 s_n^{-1}} \\ x_{s_2 s_1^{-1}} & x_{s_2 s_2^{-1}} & \dots & x_{s_2 s_n^{-1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_{s_n s_1^{-1}} & x_{s_n s_2^{-1}} & \dots & x_{s_n s_n^{-1}} \end{vmatrix}$$

If we take for  $G$  the symmetric group of degree 3 and put  $s_1 \equiv 1, s_2 \equiv (abc), s_3 \equiv (acb), s_4 \equiv (ab), s_5 \equiv (bc), s_6 \equiv (ac)$ , then

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1} & x_{s_3} & x_{s_2} & x_{s_4} & x_{s_5} & x_{s_6} \\ x_{s_2} & x_{s_1} & x_{s_3} & x_{s_6} & x_{s_6} & x_{s_4} \\ x_{s_3} & x_{s_2} & x_{s_1} & x_{s_5} & x_{s_4} & x_{s_6} \\ x_{s_4} & x_{s_6} & x_{s_5} & x_{s_1} & x_{s_3} & x_{s_2} \\ x_{s_5} & x_{s_4} & x_{s_4} & x_{s_2} & x_{s_1} & x_{s_3} \\ x_{s_6} & x_{s_4} & x_{s_6} & x_{s_3} & x_{s_2} & x_{s_1} \end{vmatrix}$$

Let  $H$ , of order  $h$ , be any subgroup of  $G$ , and arrange the elements of the first column of  $\Theta(x)$  with respect to the divisions  $s_iH$ . Then  $\Theta(x)$  can be divided up into squares of  $h$  elements each, such that in any square the sums of the elements of the  $h$  rows are the same. For, the subscripts of the elements in the square containing the  $i$ th row and the  $j$ th column are contained in  $s_iH \cdot Hs_j^{-1}$ , the subscripts for the different rows being obtained by taking the different operators in the first  $H$ , and those for the different columns by taking the different operators in the second  $H$ .

We now form the determinant  $\Theta_1$ , of degree  $n/h$ , by taking for each element the sum of the elements in any row of the corresponding square of  $\Theta(x)$ . If we arrange the elements of the determinant of the symmetric group of degree 3 with respect to the subgroup  $[1, (ab)]$ , the corresponding  $\Theta_1$  is

$$\begin{vmatrix} x_{s_1} + x_{s_4} & x_{s_3} + x_{s_5} & x_{s_2} + x_{s_6} \\ x_{s_2} + x_{s_5} & x_{s_1} + x_{s_6} & x_{s_3} + x_{s_4} \\ x_{s_3} + x_{s_6} & x_{s_2} + x_{s_4} & x_{s_1} + x_{s_5} \end{vmatrix}$$

For convenience I shall hereafter omit the  $x$ 's and write only the subscripts.

I wish to prove that  $\Theta_1$  is a factor of  $\Theta$ . (a) Add the elements of the first  $h$  rows of  $\Theta$  to the corresponding elements of the first row; the elements of the second  $h$  rows to the corresponding elements of the  $(h+1)$ th row; and so on. (b) Now so re-arrange the rows and columns that the minor formed by the first  $h$  rows and columns is  $\Theta_1$ . (c) Then multiply the elements of every row, except the first  $h$  rows, by  $h$ . (d) After this, subtract the elements of the first row from the corresponding elements of each row that was originally among the first  $h$  rows; the elements of the second row from the corresponding elements of each row that was originally among the second  $h$  rows; and so on. (e) Now to the elements of the first column add the corresponding elements of the columns that were originally the first  $h$  columns; to the elements of the second column add the corresponding elements of the columns that were originally among the second  $h$  columns; and so on. In the final form of the determinant each element of the first diagonal square of order  $h$  is  $h$  times the corresponding element of  $\Theta_1$  and all the other elements of the first  $h$  columns are zero. Moreover, a further obvious modification shows that the remaining elements of the first  $h$  rows can all be made zero. Therefore  $\Theta_1$  is a factor of  $\Theta$ .

I shall illustrate the steps of this proof by the determinant of the symmetric group of degree 3.

$$\Theta \equiv \begin{vmatrix} s_1 & s_4 & s_3 & s_5 & s_2 & s_6 \\ s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\ s_2 & s_5 & s_1 & s_6 & s_3 & s_4 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3 & s_6 & s_2 & s_4 & s_1 & s_5 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix}$$

$$(a) \begin{vmatrix} s_1 + s_4 & s_4 + s_1 & s_3 + s_6 & s_5 + s_3 & s_2 + s_6 & s_6 + s_2 \\ s_4 & s_1 & s_6 & s_3 & s_6 & s_2 \\ s_2 + s_5 & s_5 + s_2 & s_1 + s_6 & s_6 + s_1 & s_3 + s_4 & s_4 + s_3 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3 + s_6 & s_6 + s_3 & s_2 + s_4 & s_4 + s_2 & s_1 + s_5 & s_5 + s_1 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix}$$

$$(b), (c) \begin{vmatrix} s_1 + s_4 & s_3 + s_5 & s_2 + s_6 & s_4 + s_1 & s_5 + s_3 & s_6 + s_2 \\ s_2 + s_5 & s_1 + s_6 & s_3 + s_4 & s_5 + s_2 & s_6 + s_1 & s_4 + s_3 \\ s_3 + s_6 & s_2 + s_4 & s_1 + s_5 & s_6 + s_3 & s_4 + s_2 & s_5 + s_1 \\ 2s_4 & 2s_5 & 2s_6 & 2s_1 & 2s_3 & 2s_2 \\ 2s_5 & 2s_6 & 2s_4 & 2s_2 & 2s_1 & 2s_3 \\ 2s_6 & 2s_4 & 2s_5 & 2s_3 & 2s_2 & 2s_1 \end{vmatrix}$$

$$(d) \begin{vmatrix} s_1 + s_4 & s_3 + s_5 & s_2 + s_6 & s_4 + s_1 & s_5 + s_3 & s_6 + s_2 \\ s_2 + s_5 & s_1 + s_6 & s_3 + s_4 & s_5 + s_2 & s_6 + s_1 & s_4 + s_3 \\ s_3 + s_6 & s_2 + s_4 & s_1 + s_5 & s_6 + s_3 & s_4 + s_2 & s_5 + s_1 \\ s_4 - s_1 & s_5 - s_3 & s_6 - s_2 & s_1 - s_4 & s_3 - s_5 & s_2 - s_6 \\ s_5 - s_2 & s_6 - s_1 & s_4 - s_3 & s_2 - s_5 & s_1 - s_6 & s_3 - s_4 \\ s_6 - s_3 & s_4 - s_2 & s_5 - s_1 & s_3 - s_6 & s_2 - s_4 & s_1 - s_5 \end{vmatrix}$$

$$(e) \begin{vmatrix} 2(s_1 + s_4) & 2(s_3 + s_5) & 2(s_2 + s_6) & s_4 + s_1 & s_5 + s_3 & s_6 + s_2 \\ 2(s_2 + s_5) & 2(s_1 + s_6) & 2(s_3 + s_4) & s_5 + s_2 & s_6 + s_1 & s_4 + s_3 \\ 2(s_3 + s_6) & 2(s_2 + s_4) & 2(s_1 + s_5) & s_6 + s_3 & s_4 + s_2 & s_5 + s_1 \\ 0 & 0 & 0 & s_1 - s_4 & s_3 - s_5 & s_2 - s_6 \\ 0 & 0 & 0 & s_2 - s_5 & s_1 - s_6 & s_3 - s_4 \\ 0 & 0 & 0 & s_3 - s_6 & s_2 - s_4 & s_1 - s_5 \end{vmatrix}$$

If  $H'$  is conjugate to  $H$  in  $G$ , the corresponding factor  $\Theta'_1$  is the same as  $\Theta_1$ . For, the element in the  $i$ th row and  $j$ th column of  $\Theta_1$  is  $\sum s_a$ , where  $s_a$  runs through the operators of  $s_i H s_j^{-1}$ . If  $s_r H s_r^{-1} = H'$ , the element in the  $i$ th row and  $j$ th column of  $\Theta_1$  is the same as the element in the  $k$ th row and  $l$ th column of  $\Theta'_1$ , where  $s_i = s_k s_r$ ,  $s_j^{-1} = s_r^{-1} s_l^{-1}$ , since  $s_i H s_j^{-1} = s_k s_r H s_r^{-1} s_l^{-1} = s_k H' s_l^{-1}$ . Moreover,  $s_j$  fixes  $s_l$ , and therefore the elements of the  $j$ th column of  $\Theta_1$  are the same, except as to arrangement, as those of the  $l$ th column of  $\Theta'_1$ . Likewise the elements of the  $i$ th row of  $\Theta_1$  are the same, except as to arrangement, as those of the  $k$ th row of  $\Theta'_1$ . Therefore  $\Theta'_1 = \pm \Theta_1$ .